Also solved by E. H. Clarke, C. E. Horne, and A. Pelletier.

2749 [1919, 72]. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/(3 \sin^3 2\phi)$ where ϕ is the angle between one of the normals and the x-axis.

Solution by H. M. Roeser, Bureau of Standards, Washington, D. C.

The tangents to a parabola at the extremities of a focal chord intersect on the directrix at right angles. (Tanner and Allen, *Analytic Geometry*, page 227.) The tangents and normals will, therefore, form a rectangle of which the focal chord is a diagonal and whose area is equal to the product of the lengths of the tangents from their intersection on the directrix to the points of tangency. The area sought is the area of one of the triangular halves of the rectangle plus two-thirds of the area of the other triangle or five-sixths of the area of the rectangle.

Let m = slope of one of the normals. Then $y = mx - 2am - am^3$ is the equation of one normal and $y = -x/m + 2a/m + a/m_0$ is the equation of the other normal. $y = -x/m - am$ is the equation of one tangent, and $y = mx + a/m$ is the equation of the other tangent. The tangents intersect at the point $(x, y) \equiv [-a, a(1 - m^2)/m]$ and touch the curve at $(x, y) \equiv [am^2, -2am]$ and $(x, y) \equiv [a/m^2, 2a/m]$, respectively.

The lengths of the tangents are $l_1 = a(1 + m^2)\sqrt{1 + m^2}/m$ and $l_2 = a(1 + m^2)\sqrt{1 + m^2}/m^2$. The area sought is therefore $5l_1l_2/6 = 5a^2(1 + m^2)^3/6m^3 = 5a^2/(6 \sin^3 \phi \cos^3 \phi) = 20a^2/(3 \sin^3 2\phi)$.

Also solved by E. H. Clarke, H. H. Downing, Polycarp Hansen, C. E. Horne, Marcia L. Latham, A. Pelletier, and the Proposer.

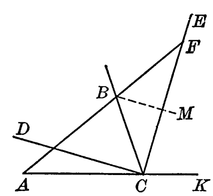
2750 [1919, 72]. Proposed by A. CAMPBELL, St. Johnsburg, Vermont.

Given the base, the sum of the sides of the triangle and the difference of the base angles, to construct the triangle.

Solution by the Proposer.

Let b, be the given base; $a + c$, the sum of the other two sides, and $\alpha = C - A$ the difference of the base angles.

On the line AK lay off $AC = b$ and at C construct an angle $ACD = \frac{1}{2}(C - A) = \frac{1}{2}\alpha$. Draw CE perpendicular to DC. With A as a center and a radius equal to $a + c$ describe an arc intersecting CE in F. Draw AF. Construct the angle BCF equal to the angle BFC. Then the triangle ABC is the required triangle.



For, triangle BCF is an isosceles triangle having its base angles equal by construction. Hence, $BC = BF$, and, therefore, $AB + BC = AF$.

Also, angle CBF = angle A + angle C, or angle MBC (BM being the bisector of angle CBF) $= \frac{1}{2}$ angle $CBF = \frac{1}{2}$(angle A + angle C) = angle BCD = angle BDC = angle A + angle DCA; whence angle $DCA = \frac{1}{2}$(angle C − angle A) $= \frac{1}{2}\alpha$.

Also solved by C. L. Arnold, George Aquis, Mary Bejsoric, P. J. da Cunha, Chang Chih-chen, H. H. Downing, A. M. Harding, C. E. Horne, Marcia L. Latham, A. Pelletier, Marian M. Torrey, and Louis Weisner.

2751 [1919, 72]. Proposed by ENOS E. WITMER, Senior in Franklin and Marshall College.

Investigate the problem of solving the equation

$$x^4 + ay^4 = w^2 + av^2. \tag{1}$$

Carmichael's *Diophantine Analysis*, Problem 18, p. 54.

SOLUTION BY THE PROPOSER.

It appears that x, y, v, w, and a are to be rational numbers and $a \neq 0$.
If $v^2 = y^4$, $w^2 = x^4$, and a solution is given by $v = \pm y^2$ $w = \pm x^2$.
If $v^2 \neq y^4$ we may proceed as follows: From the given equation, we have

$$x^4 - w^2 = a(v^2 - y^4); \tag{2}$$

whence,

$$\frac{x^4 - w^2}{v^2 - y^4} = a = \frac{(x^4 - w^2)(v^2 - y^4)}{(v^2 - y^4)^2}, \tag{3}$$

or

$$\left(\frac{x^2v \pm wy^2}{v^2 - y^4}\right)^2 - \left(\frac{x^2y^2 \pm wv}{v^2 - y^4}\right)^2 = a. \tag{4}$$

Letting

$$\frac{x^2v \pm wy^2}{v^2 - y^4} = b, \tag{5}$$

$$\frac{x^2y^2 \pm wv}{v^2 - y^4} = c \tag{6}$$

and solving (5) and (6) for x^2 and w,

$$x^2 = bv - cy^2 \tag{7}$$

and

$$w = cv - by^2. \tag{8}$$

From (7),

$$v = \frac{x^2 + cy^2}{b}, \tag{9}$$

Substituting the value of v from (9) in (8), we have

$$w = \frac{cx^2 + c^2y^2 - b^2y^2}{b}.$$

But from (4), (5), and (6), $b^2 - c^2 = a$. Hence if we put $b - c = m$, $b + c = a/m$; then

$$b = \frac{1}{2}\left(\frac{a}{m} + m\right), \qquad c = \frac{1}{2}\left(\frac{a}{m} - m\right).$$

Hence

$$x = r, \qquad y = s,$$

$$v = \frac{2r^2 + \left(\frac{a}{m} - m\right)s^2}{\frac{a}{m} + m} = \frac{2mr^2 + (a - m^2)s^2}{a + m^2},$$

$$w = \frac{\left(\frac{a}{m} - m\right)r^2 - 2as^2}{\frac{a}{m} + m} = \frac{(a - m^2)r^2 - 2ams^2}{a + m^2}.$$

2765 [1919, 171]. Proposed by A. M. HARDING, University of Arkansas.

ABC is an equilateral triangle. A point D is taken in BC such that BD is $\frac{1}{3}$ of BC and E is taken in CA such that CE is $\frac{1}{3}$ of CA. If the lines AD and BE intersect at O, show that OC is perpendicular to AD.

SOLUTION BY THE LATE L. G. WELD.

Since CD is twice CE and $\angle DCE = 60°$ the auxiliary line DE is perpendicular to CE; whence the point E is in the circumference of a circle described upon CD as a diameter. Since the triangles BCE and BOD are similar

$$BD \cdot BC = BO \cdot BE.$$

Hence, O, as well as E, lies in the circumference of the above circle and the angle COD, being inscribed in a semicircle, is a right angle.